

Super-Group Field Cosmology

Mir Faizal

Mathematical Institute, University of Oxford
Oxford OX1 3LB, United Kingdom

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Abstract

In this paper we construct a model for group field cosmology. The classical equations of motion for the non-interactive part of this model generate the Hamiltonian constraint of loop quantum gravity for a homogeneous isotropic universe filled with a scalar matter field. The interactions represent topology changing processes that occurs due to joining and splitting of universes. These universes in the multiverse are assumed to obey both bosonic and fermionic statistics, and so a supersymmetric multiverse is constructed using superspace formalism. We also introduce gauge symmetry in this model. The supersymmetry and gauge symmetry are introduced at the level of third quantized fields, and not the second quantized ones. This is the first time that supersymmetry has been discussed at the level of third quantized fields.

1 Introduction

Loop quantum gravity is an background independent way to quantize gravity, where the Hamiltonian constraint is written in terms of Ashtekar-Barbero connection and densitized triad [1]-[9]. In loop quantum gravity the curvature of the connection is expressed through the holonomy around a loop. The area of such a loop cannot take arbitrary small nonzero values because the area operator in loop quantum gravity has a discrete spectrum. At a kinematics level the Hilbert space of loop quantum gravity is a space of spin networks [10]-[13], and the time evolution of these spin networks naturally leads to the spin foam model [14]-[18]. Even though in loop quantum gravity the geometry is dynamical, the topology is fixed. This is because, like the canonical formalism, loop quantum gravity is a second quantized formalism. Just like the processes where the particle number is not conserved cannot be fully analysed in the first quantization formalism, and we have to resort to second quantization to analyse such processes; the topology changing processes cannot be fully analysed in second quantized formalism, and we have to resort to third quantization to analyse such processes [19]-[22]. A third quantization of loop quantum gravity naturally leads to group fields theory, which is a field theory defined on a group manifold [23]-[26]. Topology changing process can be analysed in the framework of group field theory. These processes occur due to the creation and annihilation of spin states. So, the Feynman amplitudes of group field theory can be equivalently

expressed in terms of spin-foam models. Many models for loop quantum cosmology have been recently analysed and are better understood than full loop quantum gravity [27]-[31]. Similarly, group field cosmology has been studied as a mini-superspace approximation to the full group field theory [32]-[34]. In group field cosmology first the Hamiltonian constraint of loop quantum cosmology is viewed as the kinetic part of the group field theory and then interactions are added. The bosonic group field cosmology has already been studied and there is no reason to restrict the statistics of the group field cosmology to bosonic case. Hence, in this paper we will also analyse a fermionic group field cosmology. We will in fact construct a supersymmetric group field cosmology in $\mathcal{N} = 1$ superspace formalism with gauge symmetry. Even though color has already been incorporated in group field theory [35]-[36], it has not been so far been incorporated in any model of group field cosmology. Furthermore, the supersymmetric loop quantum gravity have been studied only at the second quantized level [37]-[40], and supersymmetry has never been studied at the level of third quantized loop quantum gravity or loop quantum cosmology. In fact, no third quantized theory, including group field theory, has ever been supersymmetrized. Supersymmetry has been an important ingredient in the study of M-theory [41]-[43] and many phenomenological models beyond the standard model are based on it [44]-[45]. It also provides the an interesting candidate for dark matter [46]-[47]. Hence, it will be interesting to incorporate it in group field cosmology at a third quantized level. Motivated from this fact we also construct a supersymmetric group field theory with gauge symmetry in this paper.

2 Loop Quantum Cosmology

In loop quantum cosmology, the integration is restricted to a fixed three dimensional cell of finite volume \mathcal{V}_0 . The integration is restricted in order to regulate the infinities coming from homogeneous fields being integrated over a non-compact space-time [48]-[49]. The main variables in loop quantum gravity are Ashtekar-Barbero connection A_a^i and E_i^a . In the loop quantum gravity the holonomies of the connection are defined as operators. We will analyse the loop quantum cosmology for a massless scalar field ϕ in a spatially flat, homogeneous, and isotropic universe with metric,

$$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ab}dx^a dx^b. \quad (1)$$

Here $a(t)$ is the scalar factor and $N(t)$ is the lapse function. Now $A_a^i = \gamma(\omega_0^i)_a$, where γ is the Barbero-Immirzi parameter and $(\omega_0^i)_a$ is the Levi-Civita connection. In loop quantum cosmology A_a^i only depends on $c = \pm\mathcal{V}_0^{1/3}N^{-1}\gamma a'$, with a' being the time derivative of a . Furthermore, the variable promoted to operators are $p = \pm a^2\mathcal{V}_0^{2/3}$, and $\exp(i\mu c)$, where c is conjugate to p , and μ is a function of p . In the improved dynamics scheme [59], $\mu = |p|^{-1/2}$, and this forms a basis of the eigenstates of the volume operator \mathcal{V} with

$$\mathcal{V}|\nu\rangle = 2\pi\gamma G|\nu||\nu\rangle, \quad (2)$$

where $\nu = \pm a^2\mathcal{V}_0/2\pi\gamma G$ has the dimensions of length. The variable conjugate to ν is b and so the main variable for the gravitational sector are ν and $\exp(i\lambda b)$, where λ is a constant. For the matter sector one choses the usual Schroedinger

quantization scheme and the tensor product of the kinematic Hilbert space of the gravitational sector with the kinematic Hilbert space of the matter sector give the total Hilbert space of the theory.

For the dynamics of the theory the curvature of A_a^i is expressed through the holonomy around a loop. The area of such a loop cannot be smaller than a fixed minimum area because the smallest eigenvalue of the area operator in loop quantum gravity is nonzero. In Planck units the Hamiltonian constrain for a homogeneous isotropic universe with a massless scalar field ϕ , can thus be written as

$$K^2\Phi(\nu, \phi) = [E^2 - \partial_\phi^2]\Phi(\nu, \phi) = 0, \quad (3)$$

where we have defined E^2 to be

$$\begin{aligned} E^2\Phi(\nu, \phi) = & -[B(\nu)]^{-1}C^+(\nu)\Phi(\nu+4, \phi) - [B(\nu)]^{-1}C^0(\nu)\Phi(\nu, \phi) \\ & -[B(\nu)]^{-1}C^-(\nu)\Phi(\nu-4, \phi). \end{aligned} \quad (4)$$

Here we have set $\nu_0 = 4$. For the usual choice of gauge in loop quantum cosmology, we have

$$C^-(\nu) = C^+(\nu-4), \quad (5)$$

and so we need to only specify the functions $C^+(\nu)$, $C^0(\nu)$ and $B(\nu)$. These functions depend of the choice of the lapse function and the quantization scheme choose, for example with $N = 1$ in improved dynamic scheme, we have [59]

$$\begin{aligned} C^+(\nu) &= \frac{1}{12\gamma\sqrt{2\sqrt{3}}}|\nu+2| \left| |\nu+1| - |\nu+3| \right|, \\ C^0(\nu) &= -C^+(\nu) - C^+(\nu-4), \\ B(\nu) &= \frac{3\sqrt{2}}{8\sqrt{\sqrt{3}}\pi\gamma G}|\nu| \left| |\nu+1|^{\frac{1}{3}} - |\nu-1|^{\frac{1}{3}} \right|^3. \end{aligned} \quad (6)$$

Similarly, in solvable loop quantum cosmology with $N = a^3$, we have [60]

$$\begin{aligned} C^+(\nu) &= \frac{\sqrt{3}}{8\gamma}(\nu+2), \quad C^0(\nu) = -\frac{\sqrt{3}}{4\gamma}\nu, \\ B(\nu) &= \frac{1}{\nu}. \end{aligned} \quad (7)$$

There is also a super-selection for subspace of the form $|\nu+4n\rangle$, for all integers n and some state $|\nu\rangle$. It may also be noted that the matter variable appear as the time variable in loop quantum cosmology.

3 Group Field Cosmology

Just like the wave function of first quantized theories is viewed as a classical field in second quantized formalism, the wave function of the second quantized theory should be viewed as a classical field in third quantized formalism. Hence, the wave function of loop quantum cosmology will now be viewed as the classical field of group field cosmology. So, we want a free bosonic field theory such that its classical equation of motion reproduce the Hamiltonian constraint for

loop quantum gravity. The classical action for this bosonic field theory can be written as [32]

$$S_b = \sum_{\nu} \int d\phi \Phi(\nu, \phi) K^2 \Phi(\nu, \phi). \quad (8)$$

As K^2 is not diagonal in ν , it is useful to expand $\Phi(\nu, \phi)$ as

$$\begin{aligned} \Phi(b, \phi) &= \sum_{\nu} \Phi(\nu, \phi) \exp(i\nu b), \\ \Phi(\nu, \phi) &= \frac{2}{\pi} \int_0^{\pi/2} db \Phi(b, \phi) \exp(-i\nu b). \end{aligned} \quad (9)$$

Now the bosonic action is given by

$$S_b = \frac{1}{\pi} \int d\phi \int db \bar{\Phi}(b, \phi) \tilde{K}^2 \Phi(b, \phi), \quad (10)$$

where $\Phi(v, \phi) = \bar{\Phi}(\pi/2 - b, \phi)$ and the inverse of \tilde{K}^2 is the propagator of the theory. To construct the Fock space, we expand these fields into modes which we promote to operators [32]. Each of these will create or annihilate universes. The vacuum state is defined as the state annihilated by all annihilation operator, $a_k|0\rangle = 0$, and the Fock space is constructed as by the action of creation operators on this vacuum state. The vacuum state here corresponds to a state with no geometry, matter field and topological structure. The topology, geometry and matter is created by the action of creation operators on this vacuum state. Interactions will now correspond to an interaction of these universes. Hence, they will represent the process of splitting of or joining of universes. Thus the topology changing processes can be studied by including interaction terms to this bosonic free field theory. Each interaction will also correspond to a conservation of a different quantity during the process of interaction of these universes. So, for example the interaction potential of the form $\lambda\Phi^3(\nu, \phi)/3$, will correspond to a conservation of both b and p_ϕ , because it will implement locality in both the scalar factor and the scalar field.

There is no reason to assume that the statistics of the theory are bosonic and so we can also construct a fermionic model for group field cosmology. The choice of fermionic statics will change how distribution of different universes in the multiverse. In fact, attempts to explain the cosmological constant from a bosonic distributions of universes in the multiverse have led to a wrong value for the cosmological constant [50]. It will be interesting to perform a similar analysis with a fermionic distribution of universes and try to calculate the value of the cosmological constant. This is one of the motivations for studying a fermionic distribution of universes in the multiverse.

Fermionic statics will not change the dynamics of a single universe because the fermionic theory will also be required to generate the Hamiltonian constraint of loop quantum gravity. Thus, in analogy with a fermionic field in two dimensions [51], we define a fermionic field in group field cosmology as $\Psi_a(\nu, \phi) = (\Psi_1(\nu, \phi), \Psi_2(\nu, \phi))$. The spinor indices are raised and lowered by the second-rank antisymmetric tensors C^{ab} and C_{ab} , respectively,

$$\begin{aligned} \Psi^a(\nu, \phi) &= C^{ab} \Psi_b(\nu, \phi), \\ \Psi_a(\nu, \phi) &= \Psi^b(\nu, \phi) C_{ba}. \end{aligned} \quad (11)$$

These second-rank antisymmetric tensors also satisfy $C_{ab}C^{cb} = \delta_a^c$. So, we have $(\gamma^\mu)_{ab} = (\gamma^\mu)_a^c C_{cb} = (\gamma^\mu)_{ba}$ [52]. Now we define $K_\mu = (E, \partial_\phi)$ and $\eta^{\mu\nu} = (1, -1)$. Here K_μ acts like ∂_μ of a ordinary two dimensional field theory, so in analogy with $\partial_{ab} = (\gamma^\mu \partial_\mu)_{ab}$ and $\partial^2 = \partial^{ab} \partial_{ab}/2$ [51], we also define $K_{ab} = (\gamma^\mu)_{ab} K_\mu$ and

$$\begin{aligned} K^2 &= \frac{1}{2} K^{ab} K_{ab} \\ &= \frac{1}{2} (\gamma^\mu)^{ab} K_\mu (\gamma^\nu)_{ab} K_\nu \\ &= \eta^{\mu\nu} K_\mu K_\nu \\ &= [E^2 - \partial_\phi^2], \end{aligned} \tag{12}$$

where we have used $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Now we can write the action for the fermionic group field cosmology in analogy with the regular fermionic action in two dimensional spinor formalism by using the correspondence between K_μ and ∂_μ [51],

$$\begin{aligned} S_f &= \sum_\nu \int d\phi C^{bc} \Psi_c(\nu, \phi) (\gamma^\mu)_b^a K_\mu \Psi_a(\nu, \phi) \\ &= \sum_\nu \int d\phi \Psi^b(\nu, \phi) K_b^a \Psi_a(\nu, \phi). \end{aligned} \tag{13}$$

The classical equations of motion obtained from this action are

$$K_a^b \Psi_b(\nu, \phi) = 0. \tag{14}$$

Now action on it by K_b^c , and using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, we get

$$K^2 \Psi_c(\nu, \phi) = 0. \tag{15}$$

We can also construct the Fock space for this fermionic theory by first expand these fields into modes and then promoting those modes to operators. Just like the bosonic theory, the vacuum state of the fermionic theory will also be annihilated by all annihilation operators $a_k|0\rangle = 0$, and the Fock space will be constructed by the action of creation operators on this vacuum state. This vacuum state again corresponds to a state with no geometry, matter field and topological structure. Here the again modes denoted by k correspond to the momentum p_ϕ , which is the momentum associated with the scalar field ϕ [32]. We can also introduce interactions for this fermionic theory or a combination of both the fermionic and bosonic theories.

4 Supersymmetry

In this section we will derive a supersymmetric generalization of the group field cosmology. We will in fact construct a super-group field theory, such that the kinetic parts of its bosonic and fermionic component fields will generate the Hamiltonian constraint of loop quantum gravity. In order to do that we first introduce θ_a as two component anti-commuting parameters with odd Grassmann

parity and let $\theta^2 = \theta_a C^{ab} \theta_b / 2 = \theta^a \theta_a / 2$. Now using this Grassman variable θ_a , we define the generators of $\mathcal{N} = 1$ supersymmetry as

$$Q_a = \partial_a - K_a^b \theta_b. \quad (16)$$

The covariant derivative that commutes with this generator of supersymmetry can be written as

$$D_a = \partial_a + K_a^b \theta_b, \quad (17)$$

where

$$\partial_a = \frac{\partial}{\partial \theta^a}. \quad (18)$$

Hence, the full supersymmetric algebra between D_a and Q_a is given by

$$\begin{aligned} \{Q_a, Q_b\} &= (\partial_a - K_a^c \theta_c)(\partial_b - K_b^d \theta_d) \\ &\quad + (\partial_b - K_b^d \theta_d)(\partial_a - K_a^c \theta_c) \\ &= 2K_{ab}, \\ \{D_a, D_b\} &= (\partial_a + K_a^c \theta_c)(\partial_b + K_b^d \theta_d) \\ &\quad + (\partial_b + K_b^d \theta_d)(\partial_a + K_a^c \theta_c) \\ &= -2K_{ab}, \\ \{Q_a, D_b\} &= (\partial_a - K_a^c \theta_c)(\partial_b + K_b^d \theta_d) \\ &\quad + (\partial_b + K_b^d \theta_d)(\partial_a - K_a^c \theta_c) \\ &= 0, \end{aligned} \quad (19)$$

because $(\gamma^\mu K_\mu)_{ab} = (\gamma^\mu K_\mu)_a^c C_{cb} = (\gamma^\mu K_\mu)_{ba}$ [52]. Now we will write a supersymmetric theory in $\mathcal{N} = 1$ superspace formalism so that it has manifest $\mathcal{N} = 1$ supersymmetry. To do that, we define a superfield $\Omega(\nu, \phi, \theta)$, such that

$$\Omega(\nu, \phi, \theta) = \Phi(\nu, \phi) + \theta^a \Psi_a(\nu, \phi) - \theta^2 F(\nu, \phi). \quad (20)$$

Now we can write

$$\begin{aligned} \Phi(\nu, \phi) &= [\Omega(\nu, \phi, \theta)]|, & \Psi_a(\nu, \phi) &= [D_a \Omega(\nu, \phi, \theta)]|, \\ F(\nu, \phi) &= [D^2 \Omega(\nu, \phi, \theta)]|, \end{aligned} \quad (21)$$

where $'|'$ means we set $\theta_a = 0$ at the end of calculation. The supersymmetric transformations δ_s generated by Q_a can also be calculated in analogy with the regular supersymmetric theory,

$$\begin{aligned} \delta_s \Phi(\nu, \phi) &= -\epsilon^a(\nu, \phi) \Psi_a(\nu, \phi), \\ \delta_s \Psi_a(\nu, \phi) &= -\epsilon^b(\nu, \phi) [C_{ab} F(\nu, \phi) + K_{ab} \Phi(\nu, \phi)], \\ \delta_s F(\nu, \phi) &= -\epsilon^a(\nu, \phi) K_a^b \Psi_b(\nu, \phi). \end{aligned} \quad (22)$$

These transformations are similar to the regular $\mathcal{N} = 1$ transformations in three dimensions [51], with the spacetime replaced by (ν, ϕ) and ∂_{ab} replaced by K_{ab} . Now we can write a superfield theory in this $\mathcal{N} = 1$ superspace. As the field theory will be written in $\mathcal{N} = 1$, it will have manifest supersymmetry. However, we have two operators E and ∂_ϕ , so we have to work in analogy with a two dimensional superfield theory. It is well known that $\mathcal{N} = 1$ supersymmetry in three dimensions corresponds to $\mathcal{N} = 2$ supersymmetry in two dimensions.

This is because by using the projection $P_{\pm} = (1 \pm \gamma^3)/2$, we can split the super-charge Q_a as $\epsilon^a Q_a = \epsilon^-(\nu, \phi) Q_- + \epsilon^+(\nu, \phi) Q_+$. In two dimensions these Q_+ and Q_- act as two independent super-charges [52]-[55]. So, our theory actually has $\mathcal{N} = 2$ supersymmetry. Thus, again in analogy with regular free supersymmetric theory, we write the following action,

$$\begin{aligned}
S_s &= \frac{1}{2} \sum_{\nu} \int d\phi [D^2[\Omega(\nu, \phi, \theta) D^2 \Omega(\nu, \phi, \theta)]]_I \\
&= \frac{1}{2} \sum_{\nu} \int d\phi [D^2 \Omega(\nu, \phi, \theta) D^2 \Omega(\nu, \phi, \theta) \\
&\quad + D^a \Omega(\nu, \phi, \theta) D_a D^2 \Omega(\nu, \phi, \theta) \\
&\quad + \Omega(\nu, \phi, \theta) (D^2)^2 \Omega(\nu, \phi, \theta)]]_I \\
&= \frac{1}{2} \sum_{\nu} \int d\phi [F^2(\nu, \phi) + \Psi^b(\nu, \phi) K_b^a \Psi_a(\nu, \phi) \\
&\quad + \Phi(\nu, \phi) K^2 \Phi(\nu, \phi)]. \tag{23}
\end{aligned}$$

This is the required supersymmetric model whose bosonic part and fermionic parts generated the bosonic and fermionic models of loop quantum cosmology. We can now add interactions to this supersymmetric theory,

$$\begin{aligned}
S_{si} &= \sum_{\nu} \int d\phi [D^2[f(\Omega(\nu, \phi, \theta))]]_I \\
&= \sum_{\nu} \int d\phi [f''(\Omega(\nu, \phi, \theta)) (D^a \Omega(\nu, \phi, \theta))^2 \\
&\quad + f'(\Omega(\nu, \phi, \theta)) D^2 \Omega(\nu, \phi, \theta)]_I \\
&= \sum_{\nu} \int d\phi [f''(\Phi)(\nu, \phi) \Psi^2(\nu, \phi) + f'(\Phi)(\nu, \phi) F(\nu, \phi)], \tag{24}
\end{aligned}$$

where $f(\Omega(\nu, \phi, \theta))$ is a general interaction term containing polynomial terms in $\Omega(\nu, \phi, \theta)$ and $\Psi^2(\nu, \phi) = \Psi^a(\nu, \phi) \Psi_a(\nu, \phi)/2$. For example, the superspace interaction term given by

$$f(\Omega(\nu, \phi, \theta)) = \frac{\lambda}{6} \Omega^3(\nu, \phi, \theta), \tag{25}$$

generates the following component action,

$$\begin{aligned}
S_t &= \frac{1}{2} \sum_{\nu} \int d\phi [F^2(\nu, \phi) + \Psi^b(\nu, \phi) K_b^a \Psi_a(\nu, \phi) \\
&\quad + \Phi(\nu, \phi) K^2 \Phi(\nu, \phi) + 2\lambda \Phi(\nu, \phi) \Psi^2(\nu, \phi) \\
&\quad + \lambda \Phi^2(\nu, \phi) F(\nu, \phi)], \tag{26}
\end{aligned}$$

where $S_t = S_s + S_{si}$. It may be noted that the exact nature of the group field cosmology will effect the way individual universes interact in the multiverse. As it is not possible at present to view other universes, it seems difficult to see which of these models is close to being the correct low energy description to some full quantum theory of gravity that describes the full multiverse. One way to deal with, this problem is to view these models as describing the quantum

states of our universe at Planck level and then trying to analyse it to see if it can lead to some effective phenomenological description, at energy scales that we can measure. So, it is interesting consider these as patches of the our universe that interact with one another. Such a possibility has been discussed and used to analyse cosmological perturbations [56]-[58]. This possibility has also been discussed in the framework of bosonic group field cosmology [32].

5 Gauge Symmetry

In analogy with regular particle physics, we can also incorporate gauge symmetry into the third quantized group field cosmology. In fact, in this section, we will constraint supersymmetric group field cosmology with gauge symmetry. To do that we first consider complex superfields $\Omega(\nu, \phi, \theta)$ and $\Omega^\dagger(\nu, \phi, \theta)$, which are suitably contracted with generators of a Lie algebra, $\Omega(\nu, \phi, \theta) = \Omega^A(\nu, \phi, \theta)T_A$, and $\Omega^\dagger(\nu, \phi, \theta) = \Omega^{A\dagger}(\nu, \phi, \theta)T_A$. Here T_A are Hermitian generators of a Lie algebra $[T_A, T_B] = if_{AB}^C T_C$. Now we define

$$\begin{aligned}\Omega^A(\nu, \phi, \theta) &= \Phi^A(\nu, \phi) + \theta^a \Psi_a^A(\nu, \phi) - \theta^2 F^A(\nu, \phi), \\ \Omega^{A\dagger}(\nu, \phi, \theta) &= \Phi^{A\dagger}(\nu, \phi) + \theta^a \Psi_a^{A\dagger}(\nu, \phi) - \theta^2 F^{A\dagger}(\nu, \phi), \\ \Lambda^A(\nu, \phi, \theta) &= \lambda^A(\nu, \phi) + \theta^a \mu_a^A(\nu, \phi) - \theta^2 \nu^A(\nu, \phi).\end{aligned}\quad (27)$$

These superfields transform under infinitesimal gauge transformations as

$$\begin{aligned}\delta\Omega^A(\nu, \phi, \theta) &= if_{CB}^A \Lambda^C(\nu, \phi, \theta) \Omega^B(\nu, \phi, \theta), \\ \delta\Omega^{A\dagger}(\nu, \phi, \theta) &= -if_{CB}^A \Omega^{C\dagger}(\nu, \phi, \theta) \Lambda^B(\nu, \phi, \theta).\end{aligned}\quad (28)$$

These transformations can now be written as

$$\begin{aligned}\delta\Phi^A(\nu, \phi) &= if_{CB}^A \lambda^C(\nu, \phi) \Phi^B(\nu, \phi), \\ \delta\Phi^{A\dagger}(\nu, \phi) &= -if_{CB}^A (\nu, \phi) \Phi^{C\dagger}(\nu, \phi) \lambda^{B\dagger}, \\ \delta\Psi_a^A(\nu, \phi) &= if_{CB}^A [\lambda^C(\nu, \phi) \Psi_a^B(\nu, \phi) + \mu_a^C(\nu, \phi) \Phi^B(\nu, \phi)], \\ \delta\Psi_a^{A\dagger}(\nu, \phi) &= -if_{CB}^A [\Psi_a^{C\dagger}(\nu, \phi) \lambda^{B\dagger}(\nu, \phi) + \Phi^{C\dagger}(\nu, \phi) \mu_a^{B\dagger}(\nu, \phi)], \\ \delta F^A(\nu, \phi) &= if_{CB}^A [\lambda^C(\nu, \phi) F^B(\nu, \phi) + \nu^C(\nu, \phi) \Phi^B(\nu, \phi) \\ &\quad + \mu^{Ca}(\nu, \phi) \Psi_a^B(\nu, \phi)], \\ \delta F^{A\dagger}(\nu, \phi) &= -if_{CB}^A [F^{C\dagger}(\nu, \phi) \lambda^{B\dagger}(\nu, \phi) + \Phi^{C\dagger}(\nu, \phi) \nu^{B\dagger}(\nu, \phi) \\ &\quad + \Psi^{Ca\dagger}(\nu, \phi) \mu_a^{B\dagger}(\nu, \phi)].\end{aligned}\quad (29)$$

Now the super-derivative, given by $D_a = \partial_a + K_a^b \theta_b$, of these superfields does not transform like the original superfields. But we can define a super-covariant derivative for these superfields by requiring it to transform like the original superfields. Thus, we obtain the following expression for the super-covariant derivative of these superfields

$$\begin{aligned}\nabla_a \Omega^A(\nu, \phi, \theta) &= D_a \Omega^A(\nu, \phi, \theta) - if_{CB}^A \Gamma_a^C(\nu, \phi, \theta) \Omega^B(\nu, \phi, \theta), \\ \nabla_a \Omega^{A\dagger}(\nu, \phi, \theta) &= D_a \Omega^{A\dagger}(\nu, \phi, \theta) + if_{CB}^A \Omega^{C\dagger}(\nu, \phi, \theta) \Gamma_a^B(\nu, \phi, \theta).\end{aligned}\quad (30)$$

If the spinor superfield $\Gamma_a^A(\nu, \phi, \theta)$ is made to transform under gauge transformations as

$$\delta\Gamma_a^A(\nu, \phi, \theta) = \nabla_a \Lambda^A(\nu, \phi, \theta), \quad (31)$$

then the super-covariant derivative of the scalar superfields $\Omega^A(\nu, \phi, \theta)$ and $\Omega^{A\dagger}(\nu, \phi, \theta)$ transforms under gauge transformations like the original fields,

$$\begin{aligned}\delta \nabla_a \Omega^{A\dagger}(\nu, \phi, \theta) &= i f_{CB}^A \Lambda^C(\nu, \phi, \theta) \nabla_a \Omega^{B\dagger}(\nu, \phi, \theta), \\ \delta \nabla_a \Omega^A(\nu, \phi, \theta) &= -i f_{CB}^A \nabla_a \Omega^{C\dagger}(\nu, \phi, \theta) \Lambda^B(\nu, \phi, \theta).\end{aligned}\quad (32)$$

Now we can use these superfields to write an action for supersymmetric group field cosmology with gauge symmetry. To that we first define $\Gamma_a(\nu, \phi, \theta)$ to be a matrix valued spinor superfield which is suitable contracted with generators of a Lie algebra, $\Gamma_a(\nu, \phi, \theta) = \Gamma_a^A(\nu, \phi, \theta) T_A$. Now define the components of this superfield $\Gamma_a(\nu, \phi, \theta)$ to be

$$\begin{aligned}A_a(\nu, \phi) &= [\Gamma_a(\nu, \phi, \theta)]_||, \\ X(\nu, \phi) &= -\frac{1}{2} [D^a \Gamma_a(\nu, \phi, \theta)]_||, \\ A^\mu(\nu, \phi) &= -\frac{1}{2} [D^a (\gamma^\mu)_a^b \Gamma_b(\nu, \phi, \theta)]_||, \\ X_a(\nu, \phi) &= \frac{1}{2} [D^b D_a \Gamma_b(\nu, \phi, \theta)]_||.\end{aligned}\quad (33)$$

It may be noted that the gauge field $A^\mu(\nu, \phi)$ in group field cosmology is analogy to ordinary gauge field. Thus, $\Gamma_a(\nu, \phi, \theta)$ is the supersymmetric generalization of the gauge field in group field cosmology. We also define a field strength for this supersymmetric gauge field in group field cosmology as,

$$\omega_a(\nu, \phi, \theta) = \nabla^b \nabla_a \Gamma_b(\nu, \phi, \theta). \quad (34)$$

Now we can write the action for the gauge theory as

$$\begin{aligned}S_{ga} &= \sum_\nu \int d\phi [D^2 [\Omega^\dagger(\nu, \phi, \theta) \nabla^2 \Omega(\nu, \phi, \theta) \\ &\quad + \omega^a(\nu, \phi, \theta) \omega_a(\nu, \phi, \theta)]_||.\end{aligned}\quad (35)$$

As this theory has a gauge symmetry, we need to fix a gauge before we can quantize it. Thus, in analogy with ordinary supersymmetric gauge theory [51], we chose the following gauge,

$$D^a \Gamma_a(\nu, \phi, \theta) = 0. \quad (36)$$

This can be incorporated at a quantum level by adding the following gauge fixing term to original classical action,

$$S_{gf} = \sum_\nu \int d\phi [D^2 [B(\nu, \phi, \theta) D^a \Gamma_a(\nu, \phi, \theta)]_||, \quad (37)$$

where $B(\nu, \phi, \theta) = B^A(\nu, \phi, \theta) T_A$ is a auxiliary superfield. This field can be integrated out, in the path integral formalism, to impose the gauge fixing condition. Now we can write the ghost term by gauge transforming the gauge fixing condition, replacing $\Lambda(\nu, \phi, \theta) = \Lambda^A(\nu, \phi, \theta) T_A$ by ghost superfield $C(\nu, \phi, \theta) = C^A(\nu, \phi, \theta) T_A$ and then contracted it with anti-ghost superfield $\bar{C}(\nu, \phi, \theta) = \bar{C}^A(\nu, \phi, \theta) T_A$ [61],

$$S_{gh} = \sum_\nu \int d\phi [D^2 [\bar{C}(\nu, \phi, \theta) D^a \nabla_a C(\nu, \phi, \theta)]_||. \quad (38)$$

Thus, the sum of the ghost and gauge fixing terms is given by

$$S_{gh} + S_{gf} = \sum_{\nu} \int d\phi [D^2[B(\nu, \phi, \theta)D^a\Gamma_a(\nu, \phi, \theta) + \bar{C}(\nu, \phi, \theta)D^a\nabla_a C(\nu, \phi, \theta)]]_|. \quad (39)$$

It may be noted that in analogy with ordinary supersymmetric gauge theory the a mixing of D_a and ∇_a occurs in the ghost term. Now we can write the vacuum to vacuum transition as

$$Z = \int DM \exp(iS), \quad (40)$$

where $DM = D\Omega^\dagger D\Omega DC D\bar{C} DB D\Gamma_a$ and $S = S_{ga} + S_{gh} + S_{gf}$, without setting $\theta_a = 0$. Now we can derive the propagators for this theory and the Feynman rules using standard field theory techniques. What is interesting is that this action allows for processes like a bosonic universe with the wave function $\Phi(\nu, \phi)$ or a fermionic universe with the wave function $\Psi(\nu, \phi)$ to be interact with each other via exchange of third quantized virtual gauge universe $A_\mu(\nu, \phi)$. Thus, the big bang in this model can be viewed as creation of a pair of universe by the interaction of two other universes via the exchange of some third quantized virtual gauge universe. Just like the first quantized wave function is viewed as a classical field in second quantization, the second quantized wave function can be viewed as a classical field in third quantization. Now, we know that the particle creation can only be consistently explained in the framework of a second quantized theory. From the perspective of a first quantized theory the wave function of a particle will suddenly disappear when it get annihilated. Similarly, the wave function of a particle will suddenly appear when it forms. However, from the perspective of a second quantized theory when the first quantized wave function is viewed as a classical field, the disappearance or the appearance of a particle can be easily understood in the light of an interactive field theory. Similarly, from the perspective of second quantization the wave function of the universe suddenly appears at the big bang. Similarly, the wave function of a black holes seems to disappear during the process of evaporation of a black hole. However, from a third quantized perspective, it is just the creation and annihilation of these spacetimes and matter configurations, that can be understood in the light of interactive group field cosmology.

In order to consistently analyse the supersymmetric gauge field cosmology we need to analyse the third quantized BRST symmetry of the theory. These symmetries can be use to show that various topology changing processes are unitarity from a third quantized perspective. So, we will analyse the third quantized BRST symmetry for this theory. The sum of the ghost term with the gauge fixing term can be written as a total third quantized BRST variation,

$$S_{gh} + S_{gf} = \sum_{\nu} \int d\phi s [D^2[\bar{C}(\nu, \phi, \theta)D^a\Gamma_a(\nu, \phi, \theta)]]_|, \quad (41)$$

where the third quantized BRST transformations are given by

$$\begin{aligned} s\Omega^A(\nu, \phi, \theta) &= if_{CB}^A C^C(\nu, \phi, \theta)\Omega^B(\nu, \phi, \theta), \\ s\Omega^{A\dagger}(\nu, \phi, \theta) &= -if_{CB}^A \Omega^{C\dagger}(\nu, \phi, \theta)C^B(\nu, \phi, \theta), \end{aligned}$$

$$\begin{aligned}
s C^A(\nu, \phi, \theta) &= f_{CB}^A C^{C\dagger}(\nu, \phi, \theta) C^B(\nu, \phi, \theta), \\
s \Gamma_a^A(\nu, \phi, \theta) &= \nabla_a C^A(\nu, \phi, \theta), \\
s \overline{C}^A(\nu, \phi, \theta) &= B^A(\nu, \phi, \theta), \\
s B^A(\nu, \phi, \theta) &= 0.
\end{aligned} \tag{42}$$

Here we have obtained third quantized BRST by following the procedure for obtaining the BRST transformations of any gauge field theory [62]-[63]. Thus, the third quantized BRST transformations of the gauge and matter fields is there gauge transformations with the gauge parameter replaced by the ghost superfield. The third quantized BRST transformation of the ghost superfield is given by by contacting two ghost superfields with the structure constant of the gauge group and the third quantized BRST transformation of anti-ghosts is the auxiliary superfield used to impose gauge fixing condition. Finally, The third quantized BRST transformation of the auxiliary field vanishes. These third quantized BRST transformations are nilpotent, $s^2 = 0$, and so the sum of the ghost term with the gauge fixing term is invariant under these third quantized BRST transformations. As the third quantized BRST transformations of the original third quantized classical action is only a ghost valued gauge transformations, it is also invariant under these third quantized BRST transformations. Thus, the full action is invariant under these transformations, $s S = 0$. These third quantized BRST transformation can be used to construct a third quantized BRST charge and analyse the unitarity of the supersymmetric group field cosmology with gauge symmetry.

6 Conclusion

In this paper we have analysed the group field cosmology. To do so, we have analysed the loop quantum cosmology in third quantized formalism. Thus, after reviewing the basics of loop quantum cosmology, we constructed both bosonic and fermionic field theories whose classical equations of motion generated the the Hamiltonian constraint of loop quantum cosmology. We discussed the effect of interactions for these models of group field cosmology. We also constructed a supersymmetric theory in $\mathcal{N} = 1$ superspace formalism with gauge symmetry. The component fields of this supersymmetric gauge theory satisfied the Hamiltonian constraint of loop quantum cosmology. We analysed the interactions in this supersymmetric theory and derived the corresponding interactions for the bosonic and fermionic fields component fields theories. In the supersymmetric gauge theory, two real universes interacted by an exchange of virtual universes. The big bang in this model was viewed as creation of a pair of universe by the interaction of two other universes via the exchange of some third quantized virtual gauge universe. We also analysed the third quantized BRST of this supersymmetric group field theory with gauge symmetry. We stress the fact that we did not supersymmetrize the theory at the level of second quantized fields but at the level of third quantized fields. This is the first time that supersymmetry has been studied at the level of a third quantized field theory.

It would be interesting to carry to analyse further. One direction to carry out this analyses would be to perform it for the full group field theory. Both the fermionic and bosonic group field theory have been studied and color has

also been incorporated in group field theory [35]-[36]. It would thus be interesting to analyse a supersymmetric group field theory with gauge symmetry. We will be able to develop that theory in analogy with the usual gauge theory. It will be possible to derive this theory in superspace formalism with manifest supersymmetry. This analysis can also be initially done in the $\mathcal{N} = 1$ superspace formalism. however, it will be nice to analyse this present theory or the full group theory in the extended superspace formalism, with higher amount of manifest supersymmetry. It would also be interesting to analyse string field theory via group field cosmology. String theory can be viewed as two dimensional gravity coupled to matter fields. Thus, it can be quantized via loop quantum gravity. After that a group field cosmology of string theory can be constructed. This way we will be able to incorporate higher order string interactions in string field theory. In fact, the Hamiltonian constraint for string in loop quantum gravity has already been analysed [64]. Now this Hamiltonian constraint string in loop quantum gravity, $K_{string}^2 \Phi = 0$, can view this as a kinetic part of a group field theory action, $\Phi K_{string}^2 \Phi$. Then interactions can and introduce through polynomial interaction Φ^n . Hence, we will be able to go beyond the regular cubic string field theory by using group field cosmology [65]. In fact, we can use all the results of this paper to analyse this string field theory. We could also try to construct the a superstring field theory of heterotic string as a group field theory with the symmetry group $E_8 \times E_8$ or $Spin(32)/Z_2$. We can then incorporate arbitrary interactions for these string field theories via the standard methods of group field theory.

References

- [1] A. Ashtekar, Phys. Rev. D36, 1587 (1987)
- [2] A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986)
- [3] C. Rovelli and L. Smolin, Nucl. Phys. B331, 80 (1990)
- [4] C. Rovelli, Quantum Gravity, Cambridge University Press, Cambridge, UK. (2007)
- [5] G. Date and G. M. Hossain, SIGMA 8, 010 (2012)
- [6] A. Ashtekar and A. Corichi, Class. Quant. Grav. 20, 4473 (2003)
- [7] A. Ashtekar, Nature Physics 2, 726 (2006)
- [8] E. R. Livine and J. Tambornino, J. Math. Phys. 53, 012503 (2012)
- [9] M. Geiller, M. L. Rey and K. Noui, Phys. Rev. D84, 044002 (2011)
- [10] C. Rovelli and L. Smolin, Phys. Rev. D53, 5743 (1995)
- [11] J. C. Baez, Adv. Math. 117, 253 (1996)
- [12] E. Bianchi, E. Magliaro and C. Perini, Phys. Rev. D82, 024012 (2010)
- [13] C. Rovelli and F. Vidotto, Phys. Rev. D81, 044038 (2010)
- [14] L. Freidel and K. Krasnov, Adv. Theor. Math. Phys. 2, 1183 (1999)

- [15] J. C. Baez, Lect. Notes. Phys. 543, 25 (2000)
- [16] A. Perez, Class. Quant. Grav. 20, R43 (2003)
- [17] M. Geiller and K. Noui, Class. Quant. Grav. 29, 135008 (2012)
- [18] V. Bonzom, Phys. Rev. D84, 024009 (2011)
- [19] A. Buonanno, M. Gasperini, M. Maggiore and C. Ungarelli, Class. Quant. Grav. 14, L97 (1997)
- [20] L. O. Pimentel and C. Mora, Phys.Lett. A280 (2001) 191-196
- [21] S. R. Perez and P. F. G. Diaz, Phys. Rev. D81, 083529 (2010) 127
- [22] V. P. Maslov and O. Y. Shvedov, Phys. Rev. D60, 105012 (1999)
- [23] A. Baratin and D. Oriti, Phys. Rev. D 85, 044003 (2012)
- [24] M. Smerlak, Class. Quant. Grav. 28, 178001 (2011)
- [25] A. Baratin, F. Girelli and D. Oriti, Phys. Rev. D83, 104051 (2011)
- [26] A. Tanasa, J. Phys. A45, 165401 (2012)
- [27] A. Ashtekar, J. Phys. Conf. Ser. 189, 012003 (2009)
- [28] L. Qin, G. Deng and Y. Ma, Commun. Theor. Phys. 57, 326 (2012)
- [29] B. Gupt and P. Singh, Phys. Rev. D85, 044011 (2012)
- [30] E. W. Ewing, Class. Quant. Grav. 29, 085005 (2012)
- [31] L. Qin and Y. Ma, Mod. Phys. Lett. A27, 1250078 (2012)
- [32] G. Calcagni, S. Gielen and D. Oriti, Class. Quantum Grav. 29, 105005 (2012)
- [33] L. A. Glinka, Grav. Cosmolo. 15, 317 (2009)
- [34] S. Gielen, J. Phys. Conf. Ser. 360, 012029 (2012)
- [35] R. Gurau, Commun. Math. Phys. 304, 69 (2011)
- [36] R. Gurau, Ann. Henri. Poincare. 11, 565 (2010)
- [37] Y. Ling and L. Smolin, Phys. Rev. D61, 044008 (2000)
- [38] R. Oeckl, J. Geom. Phys. 46, 308 (2003)
- [39] C. H. Chou, Y. Ling, C. Soo and H. L. Yu, Phys. Lett. B637, 12 (2006)
- [40] Y. Ling, J. Math. Phys. 43, 154 (2002)
- [41] I. L. Buchbinder, E. A. Ivanov, O. Lechtenfeld, N. G. Pletnev, I. B. Samsonov and B. M. Zupnik, JHEP. 0903, 096 (2009)
- [42] L. Mazzucato, Y. Oz and S. Yankielowicz, JHEP. 0711, 094 (2007)
- [43] W. Taylor, Rev. Mod. Phys. 73, 419 (2001)

- [44] M. Gozdz, Wieslaw A. Kaminski and A. Wodecki, Phys. Rev. C69, 025501 (2004)
- [45] M. Cvetič, P. Langacker and J. Wang, Phys. Rev. D68, 046002 (2003)
- [46] S. Akula, M. Liu, P. Nath and G. Peim, Phys. Lett. B709, 192 (2012)
- [47] S. Profumo, Phys. Rev. D84, 015008 (2011)
- [48] A. Ashtekar and P. Singh, Class. Quant. Grav. 28, 213001 (2011)
- [49] K. Banerjee, G. Calcagni and M. Martin-Benito, SIGMA 8, 016 (2012)
- [50] S. Coleman, Nucl. Phys. B310, 643 (1988)
- [51] S. J. Gates Jr, M. T. Grisaru, M. Rocek and W. Siegel, Front.Phys. 58, 1 (1983)
- [52] D. V. Belyaev and P. V. Nieuwenhuizen, JHEP. 0804, 008 (2008)
- [53] M. Faizal and D. J. Smith, Phys. Rev. D85, 105007 (2012)
- [54] D. S. Berman and D. C. Thompson, Nucl. Phys. B820, 503 (2009)
- [55] M. Faizal, JHEP. 1204, 017 (2012)
- [56] M. Bojowald, Gen.Rel.Grav. 38, 1771 (2006)
- [57] G. I. Rigopoulos and E. P. S. Shellard, Phys. Rev. D 68, 123518 (2003)
- [58] D. Wands, K. A. Malik, D. H. Lyth, and A. R. Liddle, Phys. Rev. D 62, 043527 (2000)
- [59] A. Ashtekar, T. Pawłowski, and P. Singh, Phys. Rev. D74, 084003 (2006)
- [60] A. Ashtekar, A. Corichi, and P. Singh, Phys. Rev. D77, 024046 (2008)
- [61] L. D. Faddeev and V. N. Popov, Phys. Lett. B25, 29 (1967)
- [62] C. Becchi, A. Rouet and R. Stora, Phys. Lett. B52, 344 (1974)
- [63] C. Becchi, A. Rouet and R. Stora, Commun. Math. Phys. 42, 127 (1975)
- [64] T. Thiemann, Class. Quant. Grav. 23, 1923 (2006)
- [65] E. Witten, Nucl. Phys. B268, 253 (1986)